

Computer exercise – Markov chains and Markov processes

Automation, EIEN50 (former EIEF45)

Goal

The purpose of these simulations is to study and analyze some fundamental properties of Markov chains and Markov processes. One is ergodicity. What does it look like when a Markov chain is ergodic or not ergodic? Another property is the interpretation of efficiency and availability, as expressed by Markov processes.

File download

For this exercise you need to download some Matlab files. The files are originally written in an old Matlab 6.1 and will work on any newer version. Download the zipped file and put it in a suitable directory on your computer (e.g. markov). The file is a zipped archive and once downloaded it should easily be extracted. If you run into problems, please contact Ulf Jeppsson (ulf.jeppsson@iea.lth.se).

Report

These exercises are carried out either **individually** or in **groups of two**. The report should contain answers to the questions stated including your own comments, motivations and conclusions. When “**(Provide plot)**” is indicated, plots should be included in the answer to that particular question. You can write the reports either in Swedish or English. Note: remember to put your name and email address on a cover page.

The reports can be submitted electronically (give the file a name of structure **AUT1_Familyname** (as a pdf-file)) or as a hard copy **not later than February 2, 2022** to ulf.jeppsson@iea.lth.se You have to draw some state graphs. It is OK to write them by hand and supplement the file with this paper if you do not want to draw the state graphs on your computer.

Feedback

In order to make a better course planning for the next time it would be greatly appreciated if you try to estimate the time you spend on this exercise. Indicate the number of hours you spent on this task on the cover page. Of course we are happy if you can also provide your own comments on the exercise to help us improve the task for next year. Feedback to you on your report will be provided by email.

Help

If you need help, send me an email: ulf.jeppsson@iea.lth.se or give me a call: 046 222 9287. I will try to answer your questions as quickly as possible.

Good luck!

Ulf Jeppsson

Markov chains and Markov processes

Theory Preparation

Read Chapter 6 (Section 6.6 not included) in the textbook.

1. Initiation

Start Matlab and make sure that you are in the directory in which you unzipped the archive. Type

```
>>what
```

to see which Matlab files are in the current directory. Otherwise, make use of the “set path” menu in Matlab. You may split up a plot in two parts or make more sophisticated plots. Use the commands `plot`, `subplot`, `figure`, `hold`,... Use the `help` commands to find more hints about plotting.

Please remember that Matlab distinguishes between small and capital letters.

Markov chains

There is an m-file called `markovchain`. The file basically performs the probability propagation equation (6.8)

$$\mathbf{p}(k+1) = \mathbf{p}(k) \mathbf{P}$$

for 10 time steps based on the data you provide (start the script by simply writing `markovchain` in the Matlab command window). It plots each state probability as a function of the time steps. It will also calculate the stationary solutions if possible (you can look in the m-file how this is done), the eigenvalues and \mathbf{P} raised to a large number.

Markov processes

There are two m-files to integrate the differential equation (6.24)

$$\frac{d}{dt} p(t) = p(t) \cdot A$$

They are called `markovprocess` and `markovprocessfunc`. By giving the Matlab command

```
>>markovprocess
```

you will start the simulation. The m-file asks for the generator \mathbf{A} and the initial condition of p (as a row vector). Note that both \mathbf{A} and p are transposed during the computation, because the integration algorithm `ode45` assumes that the vectors are column vectors. It will also calculate the stationary solutions if possible (you can look in the m-file how this is done), the eigenvalues of \mathbf{A} and $e^{A t}$ for a large t value.

For the exercise you are free to modify the m-files to provide added or modified functionality as you see fit (if you want to plot things differently, extend the time of the simulation etc.). If things do not work out as you hoped you can always download the original files again – so do not be afraid to use the full power of Matlab.

2. Markov Chain Simulations

You start the simulation of the Markov chains simply by typing
`>>markovchain`
in Matlab.

Consider a Markov chain with the following transition probability matrix

$$P = \begin{pmatrix} 0.1 & 0.4 & 0.5 \\ 0.2 & 0 & 0.8 \\ 0.4 & 0.3 & 0.3 \end{pmatrix}$$

Questions and tasks:

- 2.1 Draw the state graph.
- 2.2 Is the chain ergodic?
- 2.3 Simulate the chain for three different initial conditions, starting in either state 1, state 2 or state 3. **Plot** $p(k)$ and comment the results. *Note:* it is illustrative to put all the curves on the same plot (even if it becomes clogged!). Use the `hold` command to plot the three cases on the same plot.

Now simulate the systems below (from the text book). The purpose is to find out how $\mathbf{p}(k)$ propagates as a function of k . Start with different initial values of \mathbf{p} . Find out if the process converges or not.

$$6.3d. P = \begin{pmatrix} 0 & 0 & 1 \\ 0.5 & 0.3 & 0.2 \\ 1 & 0 & 0 \end{pmatrix}$$

$$6.3e. P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \\ 0 & 0.3 & 0 & 0.7 \end{pmatrix}$$

$$6.3f. P = \begin{pmatrix} 0.1 & 0.8 & 0.1 \\ 0.9 & 0 & 0.1 \\ 0.2 & 0.2 & 0.6 \end{pmatrix}$$

$$6.3g. P = \begin{pmatrix} 1 & 0 & 0 \\ 0.21 & 0.79 & 0 \\ 0.17 & 0.35 & 0.48 \end{pmatrix}$$

Questions and tasks:

For *each one* of the systems 6.3d,e,f,g you have to answer specifically:

- 2.4 **Plot** the values of $\mathbf{p}(0)$, $\mathbf{p}(1)$, $\mathbf{p}(2)$, $\mathbf{p}(10)$ for at least two different sets of initial values.
- 2.5 If the process is not ergodic, how do you notice that?
- 2.6 Draw a state graph and comment the results.
- 2.7 Look at the eigenvalues of \mathbf{P} . Comment the values. What is the interpretation that one eigenvalue is always = 1? Notice the eigenvalues when the process is not ergodic! Remember that it may also be useful to look at \mathbf{P} raised to the power of a large value.

For the next task, the following transition probability matrix is given:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Questions and tasks:

2.8 Simulate and **plot** $\mathbf{p}(k)$ as a function of k for some different initial conditions. Does \mathbf{p} converge to a stationary value? Is the system ergodic?

2.9 Draw a state graph and interpret the result.

3. Markov Process Simulations

You will start the simulation of the Markov processes simply by typing
>>markovprocess
in Matlab.

Two machines can be in two states each, operating (2) or under repair (1). Thus, the system can be in any of the four states,

$s_1 = (1,1)$	both machines under repair
$s_2 = (2,1)$	machine 1 operating
$s_3 = (1,2)$	machine 2 operating
$s_4 = (2,2)$	both machines operating

Repair or failure can be described by the generator matrix A . The repair rates for machine ($i = 1, 2$) is r_i and the failure rates are f_i ($i=1, 2$). Now use this system to answer questions 3.1 – 3.6.

Questions and tasks:

3.1 Draw a state graph for the system.

The probability that the system is in the state s_i is defined as p_i . Simulate how $p(t)$ is propagating for some various choices of r_i and f_i , e.g.

(1) $r_1 = 0.2$ (h^{-1}); $r_2 = 0.25$; $f_1 = 0.15$; $f_2 = 0.1$

(2) $r_1 = 0.4$ (h^{-1}); $r_2 = 0.5$; $f_1 = 0.15$; $f_2 = 0.1$

3.2 For each choice of r and f parameters, simulate two different initial conditions on the same plot. **Provide one plot** for each set of r and f parameters.

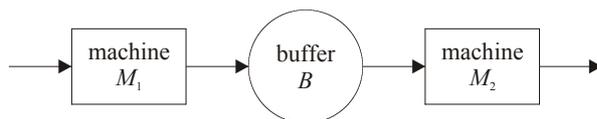
3.3 Is the process ergodic?

3.4 Note the value of $p(\bullet)$ for the first parameter set. Provide *hand calculations* to find the stationary solution to the same problem and compare with Matlab. Include the hand calculations in the report.

3.5 Check that the sum of the elements in $p(t)$ is equal to 1! (**Notice** that the Matlab m-file produces a matrix x . The rows of the matrix are the probabilities, and the last row consists of the state probabilities at the final time)

3.6 Give an interpretation of the values of the eigenvalues of A .

For the next set of questions we assume a *transfer line* consisting of 2 machines (M_1, M_2) with an intermediate buffer, according to the figure below.



The system state is described by $s(n, \bullet_1, \bullet_2)$, where \bullet_i is the state for machine i and n is the number of articles in the buffer. Assume the buffer can contain only 0 or 1 article, while a machine can be either operating ($\bullet_i = 1$) or under repair ($\bullet_i = 0$). The probability that machine i will fail during the time dt is defined by $f_i dt$, while the probability that it will be repaired during a similar time interval is $r_i dt$. The machines have the production rate μ_i .

Questions and tasks:

3.7 How many dynamical states does the system have?

3.8 Simulate the system and assume the following parameters:

$$r_1 = 0.1 \quad r_2 = 0.15$$

$$f_1 = 0.07 \quad f_2 = 0.12$$

$$\mu_1 = 0.5 \quad \mu_2 = 0.7$$

Assume that the system is initiated with both machines operating and an empty buffer in between.

3.9 Note the stationary probabilities. Which state is the most probable state?

3.10 Calculate the *system efficiency* of machine 1 (E_1), i.e. the probability that machine 1 is producing = the fraction of the total time that machine 1 is producing. Make the same calculation for machine 2 (E_2). Then calculate the overall *system productivity* $P = \mu_1 \cdot E_1 = \mu_2 \cdot E_2$.

3.11 What is the probability that M_2 is starved?

3.12 What is the probability that M_1 is blocked?

The formal part of this exercise is now completed. Hopefully you have increased your knowledge about Markov chains and Markov processes. Since you are now familiar with the short Matlab scripts you used in this exercise please continue using them to learn even more. Use examples in the text book or set up your own problems, which you can first try to solve by hand calculations and then use Matab to generate the 'correct' solutions.