Markov Processes

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Fundamentals (1)

- Transitions in discrete time –> Markov chain
- When transitions are stochastic events at arbitrary point in time –> Markov process
- Continuous time description

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Fundamentals (2)

- Consider the time interval Δt
- Probability for state transition: $a \cdot \Delta t$
- Parameter a = intensity
  (e.g. production rate, repair rate) dimension *1/time*

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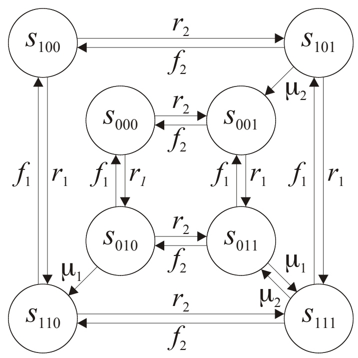
State graph



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State transition (1)

P[s_i(t)] = probability
to be in state s_i at time *t*

Probability for transition from s_i to s_j during the time Δt is $a_{ij} \cdot \Delta t$

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State transition (2)

$$P[s_j(t+\Delta t)] =$$
$$P[s_j(t)] \cdot (\text{prob no transfer from } s_j) +$$
$$+P[s_1(t)] \cdot a_{1j}\Delta t + P[s_2(t)] \cdot a_{2j}\Delta t + \ldots +$$
$$+P[s_{j-1}(t)] \cdot a_{j-1,j}\Delta t + P[s_{j+1}(t)] \cdot a_{j+1,j}\Delta t + \ldots +$$
$$+P[s_m(t)] \cdot a_{mj}\Delta t$$

(assuming a total of *m* states)

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Probability of *no* transfer

Can be expressed as:

$$1 - (\text{prob to leave the state } s_j) =$$
$$= 1 - \sum_{i \neq j}^{m} a_{ji}\Delta t$$

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State transition (3)

$$P[s_j(t+\Delta t)] =$$
$$P[s_j(t)] \cdot \left(1 - \sum_{i \neq j}^{m} a_{ji}\Delta t\right) +$$
$$+ \sum_{i \neq j}^{m} P[s_i(t)] a_{ij}\Delta t \quad (j = 1, 2, \ldots m)$$

$1 + a_{jj}\Delta t$

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State transition (4)

$$P[s_j(t+\Delta t)] =$$

$$= P[s_j(t)]\cdot\left(1 + a_{jj}\Delta t\right) +$$

$$+\sum_{i\neq j}^{m} P[s_i(t)]a_{ij}\Delta t$$

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State transition (5)

$$P[s_j(t+\Delta t)] - P[s_j(t)] =$$

$$= \sum_{i=1}^{m} P[s_i(t)]a_{ij}\Delta t$$

$$\frac{d}{dt}P[s_j(t)] = \sum_{i=1}^{m} P[s_i(t)]a_{ij}$$

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Probability vector

$$\mathbf{p}(t) = \left(p_1, p_2, \cdots, p_m\right) =$$

$$= \left(P[s_1(t)]\;\; P[s_2(t)] \cdots P[s_m(t)]\right)$$

Note: a row vector! Sum of **p**(t) = 1!

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State transition (6)

$$\frac{d}{dt}\mathbf{p}(t) = \mathbf{p}(t)\cdot\mathbf{A}$$

Solution:

$$\mathbf{p}(t) = \mathbf{p}(0)\cdot e^{\mathbf{A}t}$$

A is the *generator* or *transition* matrix
Note: Row sum of **A** = 0, one eigenvalue always = 0

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Stationary solution

$$0 = \mathbf{p}(t) \cdot \mathbf{A} = \bar{\mathbf{p}} \cdot \mathbf{A}$$

must use the extra condition: sum(**p**) = 1

Interpretation:

flow of events = rate · probability

In stationarity:

the flow of events into s_i is equal to the flow of events out of s_i

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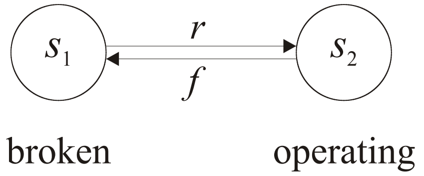
Simple example



Generator $\mathbf{A} = \begin{pmatrix} -r & r \\ f & -f \end{pmatrix}$

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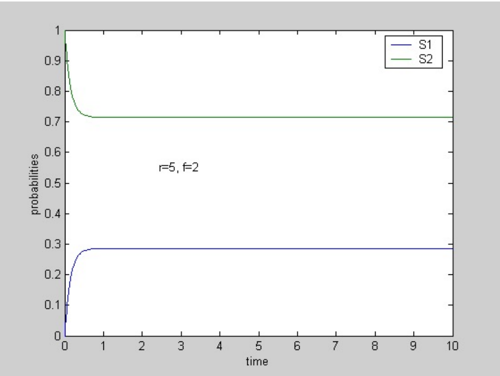
Simulation 2 state system



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Markov processes - Summary

- P[s_i(t)] = prob. to be in the state s_i at time *t*
- **p** is a row vector
- a_{ij} the transfer rate (a ”speed”)
- The generator **A**
- Sum of **p** = 1, row sum of **A** = 0
- Stationarity: **p**(t)***A** = (t -> inf) = $\mathbf{p}_{stat}$***A** = 0
- General: $\frac{d}{dt}\mathbf{p}(t) = \mathbf{p}(t) \cdot \mathbf{A}$

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The Poisson process

- Special case of the Markov process
- The prob. for an event during the time interval Δt is $\lambda \cdot \Delta t$
- *Only 1 event at a time*
- *No memory:* the number of events within $[t_1,t_2]$ independent of the number of events within $[t_3,t_4]$ (no overlap)
- Time intervals between events are independent and exponentially distributed
- Average waiting time $1/\lambda$

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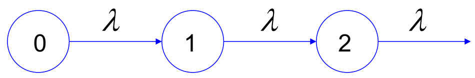
A birth process

- Describes the arrival pattern
- Arrival rate λ



$$\frac{d}{dt} p_j(t) = \lambda_{j-1} p_{j-1}(t) - \lambda_j p_j(t)$$

$$(j = 1,2,\ldots,k)$$

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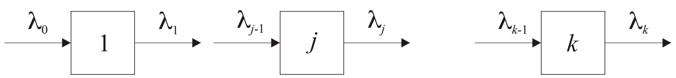
Simple transfer line – a birth process



$$\frac{d}{dt} p_j(t) = \lambda_{j-1} p_{j-1}(t) - \lambda_j p_j(t)$$

$$(j = 1,2,\ldots,k)$$

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The transfer line generator

$$\mathbf{A} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\lambda_1 & \lambda_1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -\lambda_{k-1} & \lambda_{k-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\lambda_k & \lambda_k \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

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Transfer line solution

$$p_0(t) = e^{-\lambda_0 t}$$

$$p_1(t) = \lambda t \, e^{-\lambda t}$$

$$p_j(t) = \frac{(\lambda t)^j}{j!} e^{-\lambda t} \qquad 1 \le j \le k, \quad t \ge 0$$

Special case
$\lambda_j = \lambda$

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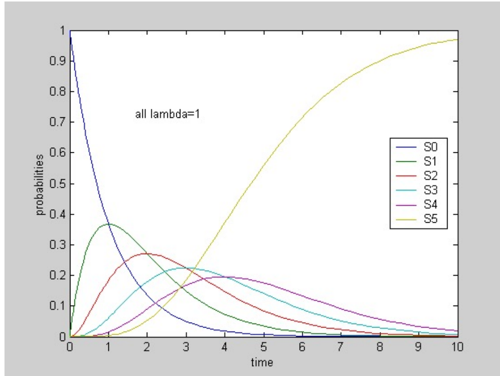
Transfer line process

Input + 4 machines + output → 6 states



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Work in progress

Generalised, birth processes can be used to model arrivals into a system. Assumption infinite set of jobs. $p_n(t)$ is the probability that exactly n jobs have arrived in the interval $[0,t]$.
At least one job arrived: $1 - p_0(t)$
More than one job: $1 - p_0(t) - p_1(t)$ etc.

Average number of jobs $[0,t]$ = work in progress = WIP=

$$\bar{L} = \sum_{n=0}^{\infty} n \cdot p_n(t) = \ldots = \lambda t$$

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Example (Question 6.16)

The arrival of articles to a process can be modelled as a Poisson process. Assume arrival rate is $\lambda = 2 \text{ min}^{-1}$. Calculate the probability that there will be more than 3 jobs in the system after 1 minute.

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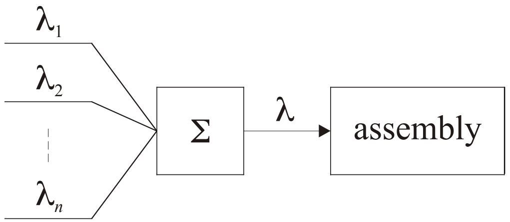
Superposition



$$\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

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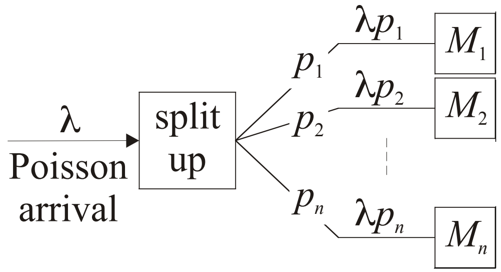
Decomposition



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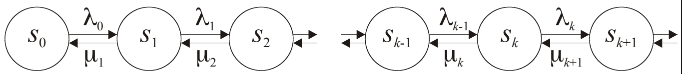
Birth-death process



$$p_{k+1} = \frac{\lambda_k}{\mu_{k+1}} p_k \qquad k = 0,1,2,\ldots$$

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Birth-death generator

$$\mathbf{A} = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots & & & & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots & & & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \cdots & & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \mu_k & -(\lambda_k+\mu_k) & \lambda_k & 0 & \cdots \\ \cdots & \cdots & \cdots & 0 & \mu_{k+1} & -(\lambda_{k+1}+\mu_{k+1}) & \lambda_{k+1} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Foundation for queuing theory (next course).
μ represents departure rate (or production rate).

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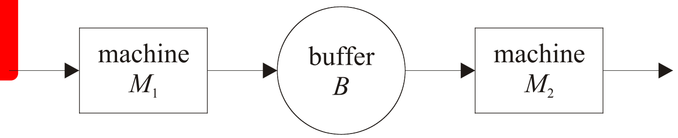
Two machines and one buffer



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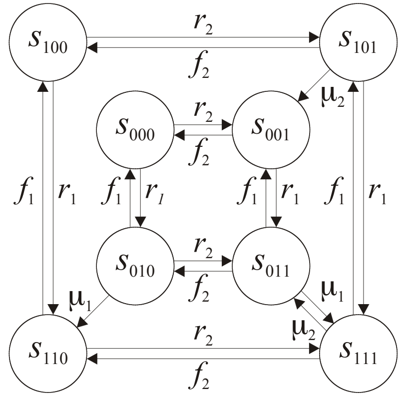
State graph



$s(n, \alpha_1, \alpha_2)$

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Probability vector

$$\mathbf{p}(t) = \left[p_{000}, p_{001}, \ldots, p_{N11}\right]$$

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Isolated machine efficiency

Isolated machine efficiency:	$e_i = \frac{r_i}{r_i + f_i}$

Isolated machine production:	$\rho_i = \mu_i \cdot e_i$

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System efficiencies

Machine 1: $E_1 = \sum_{n=0}^{N-1} \sum_{\alpha_2=0}^{1} p(n,1,\alpha_2)$

Machine 2: $E_2 = \sum_{n=1}^{N} \sum_{\alpha_1=0}^{1} p(n,\alpha_1,1)$

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Total system production

$$P_s = \mu_1 \cdot E_1 = \mu_2 \cdot E_2$$

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Average buffer size

$$\overline{n} = \sum_{n=0}^{N} \sum_{\alpha_1=0}^{1} \sum_{\alpha_2=0}^{1} n \cdot p(n,\alpha_1,\alpha_2)$$

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