Dynamic Systems

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Achievements

- 2-3 lectures on Chapter 3
- Goal: you should
  - Understand the basic concepts and ideas of different types of models
  - Formulate your own simple dynamic models based on given information
  - Analyse and simulate small dynamic systems
- No in-depth mathematics, examples during lectures
- Get an early start for the second simulation exercise

Model history

Isaac Newton in 1687:
Mathematical Principles of Natural Philosophy

Formulation of fundamental laws of force and motion

Scientific conclusion:
Nature has laws, and we (humans) can find them!

The model

A model represents a compact description of our understanding of a real world process and a way to conceptualize our knowledge

Types of models: linguistic, mental, visual, physical, mathematical etc.

Fundamental issue: the **purpose** of the model (e.g. design, research, control, forecasting, analysis, education)

The simplest model fulfilling its purpose is normally the best

Why modelling?

Too expensive not to (e.g. space travel)

Too dangerous not to (e.g. nuclear power plants)

Too time consuming not to (e.g. biological processes)

Non-existing systems (e.g. a new bridge)

Test hypotheses and gain new knowledge

Mathematical models

Static – dynamic

Linear – nonlinear

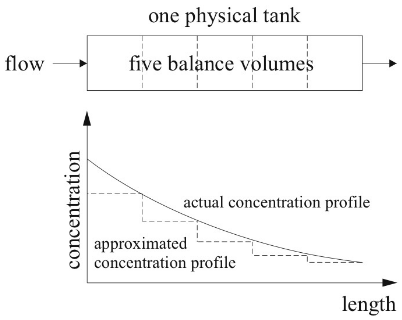
Continuous time – discrete time

Deterministic – stochastic

Internal (mechanistic, white-box) – external (black-box)

Distributed – lumped parameter

Concentration – tanks in series



Mathematical models

d**x**/dt = **A**·**x**(t) + **B**·**u**(t) (linear)

Purpose: Analysis, e.g. stability, robustness, oscillations, control design
Tools: Analytical, e.g. eigenvalues (poles), zeros, observability, controllability, reachability, Bode-plots, etc., etc.

d**x**/dt = **f**(**x**,**u**) or d**x**/dt = **f**(**x**,**u**,parameters) (nonlinear)
Purpose: scientific & detailed understanding
Tools: numerical software and computers, e.g. simulation, Monte-Carlo, sensitivity, uncertainty

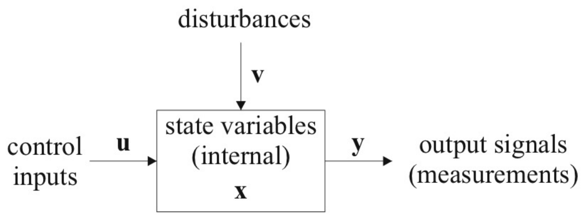
Mathematical models

The time scale is an important characterization of a dynamic system

Conservation of mass and energy must always be fulfilled!!!

A differential equation:
(the rate of change) = (rate of inflow) – (rate of outflow) + (rate generated within system) – (rate consumed within system)

A dynamical system



disturbances

v

control inputs

u

state variables (internal)

x

y

output signals (measurements)

The state equation

$$\frac{dx}{dt} = Ax + Bu \qquad y = Cx + Du$$

$$\frac{dx}{dt} = f(x,u) \qquad y = g(x,u)$$

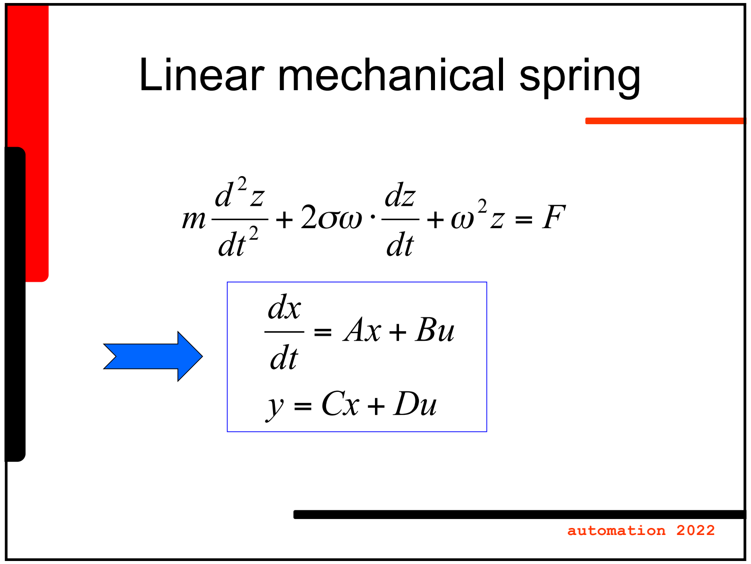
The biochemical reactor (b=0)

$$\frac{\partial S}{\partial t} = \frac{Q}{V}S_{in} - \frac{Q}{V}S - \frac{\mu_{max}}{Y}\frac{S}{K_s + S}X$$

$$\frac{\partial X}{\partial t} = \frac{Q}{V}X_{in} - \frac{Q}{V}X + \mu_{max}\frac{S}{K_s + S}X$$

Stationary solution for S

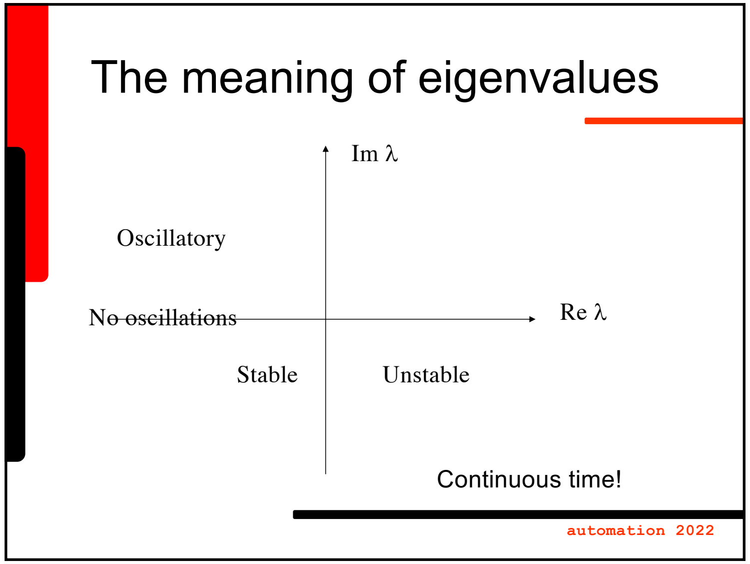
$$Y\left(-\frac{Q}{V} + \mu_{\max}\right)S^2 + \left(-\mu_{\max}X_{in} - \mu_{\max}YS_{in} - YK_S\frac{Q}{V} + Y\frac{Q}{V}S_{in}\right)S + Y\frac{Q}{V}K_S S_{in} = 0$$



Linear mechanical spring

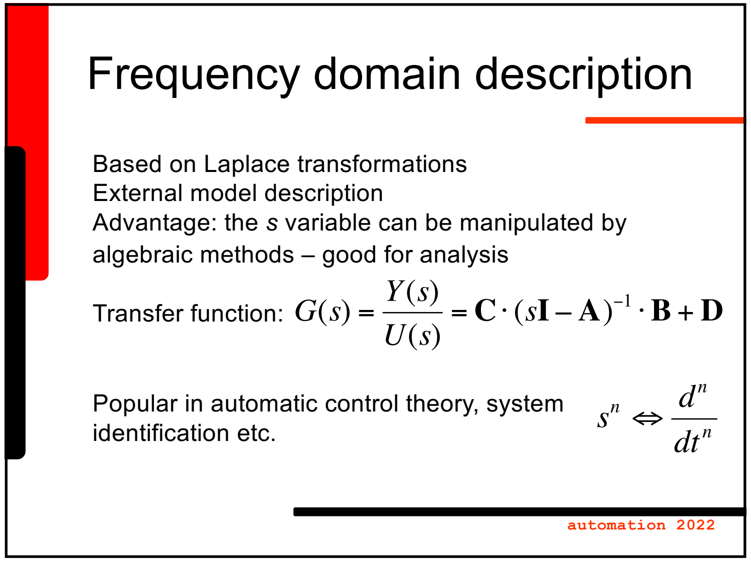
$$m\frac{d^2z}{dt^2} + 2\sigma\omega\cdot\frac{dz}{dt} + \omega^2 z = F$$

$$\Rightarrow \boxed{\begin{aligned}\frac{dx}{dt} &= Ax + Bu\\ y &= Cx + Du\end{aligned}}$$



The meaning of eigenvalues

Continuous time!

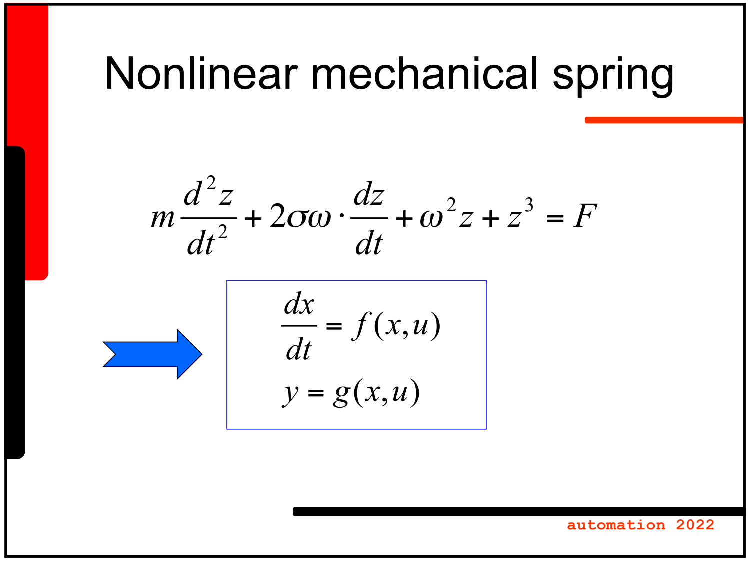


Frequency domain description

Based on Laplace transformations
External model description
Advantage: the *s* variable can be manipulated by algebraic methods – good for analysis

Transfer function: $G(s) = \dfrac{Y(s)}{U(s)} = \mathbf{C}\cdot(s\mathbf{I}-\mathbf{A})^{-1}\cdot\mathbf{B}+\mathbf{D}$

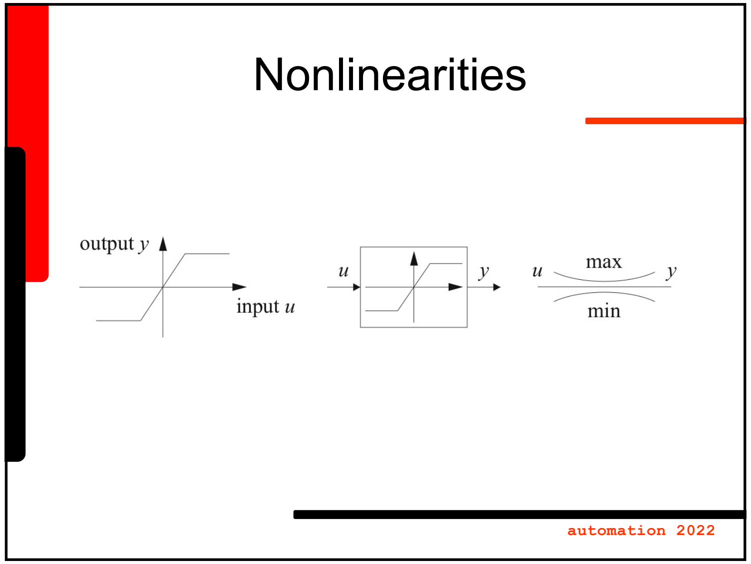
Popular in automatic control theory, system identification etc. $s^n \Leftrightarrow \dfrac{d^n}{dt^n}$



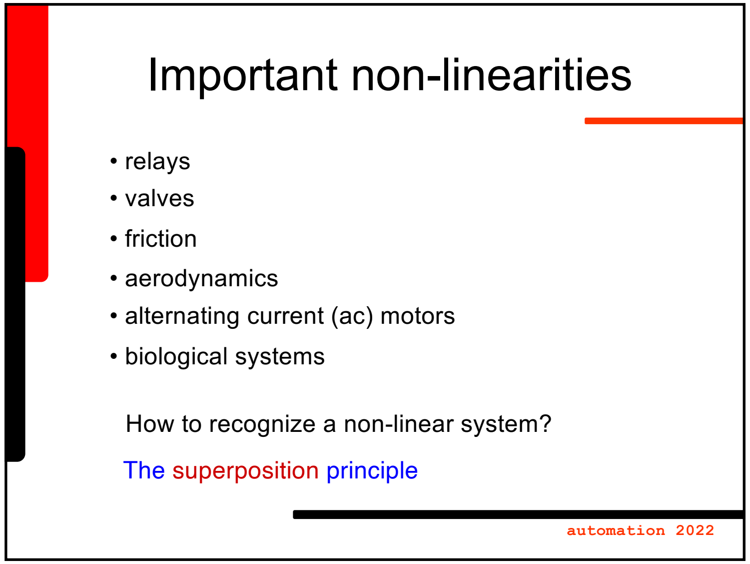
Nonlinear mechanical spring

$$m\frac{d^2z}{dt^2} + 2\sigma\omega\cdot\frac{dz}{dt} + \omega^2 z + z^3 = F$$

$$\Rightarrow \boxed{\begin{aligned}\frac{dx}{dt} &= f(x,u)\\ y &= g(x,u)\end{aligned}}$$



Nonlinearities

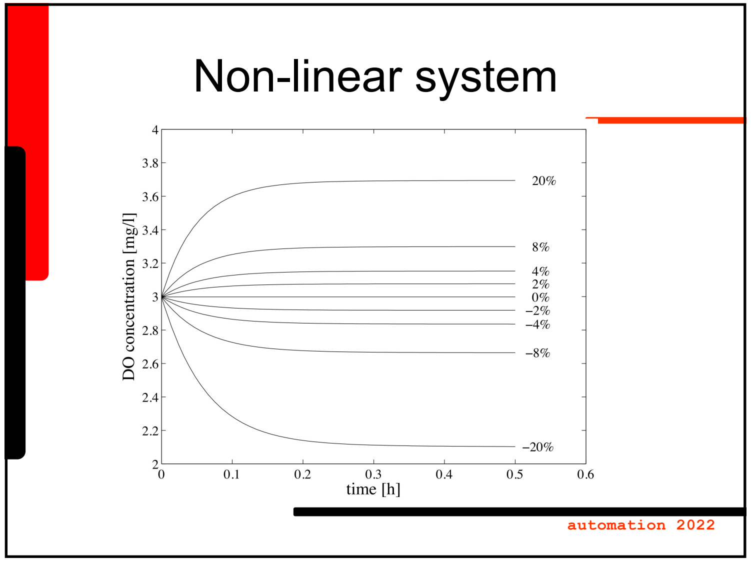


Important non-linearities

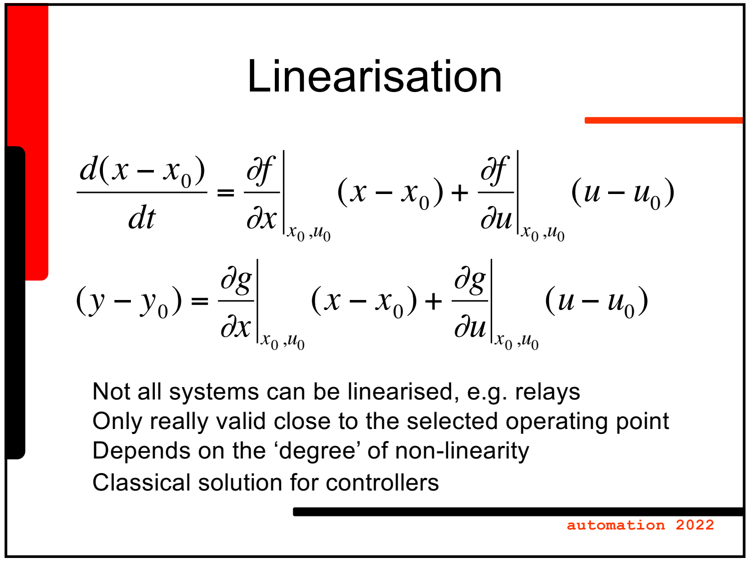
- relays
- valves
- friction
- aerodynamics
- alternating current (ac) motors
- biological systems

How to recognize a non-linear system?

The superposition principle



Non-linear system

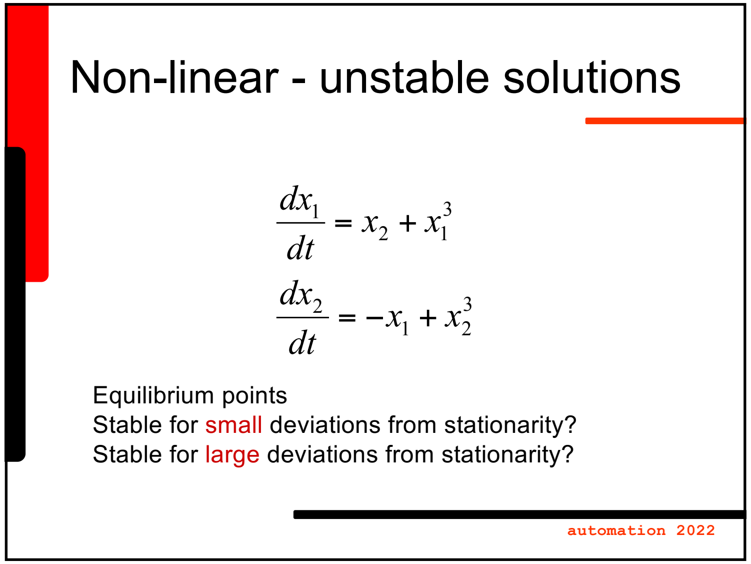


Linearisation

$$\frac{d(x-x_0)}{dt} = \left.\frac{\partial f}{\partial x}\right|_{x_0,u_0}(x-x_0) + \left.\frac{\partial f}{\partial u}\right|_{x_0,u_0}(u-u_0)$$

$$(y-y_0) = \left.\frac{\partial g}{\partial x}\right|_{x_0,u_0}(x-x_0) + \left.\frac{\partial g}{\partial u}\right|_{x_0,u_0}(u-u_0)$$

Not all systems can be linearised, e.g. relays
Only really valid close to the selected operating point
Depends on the ‘degree’ of non-linearity
Classical solution for controllers

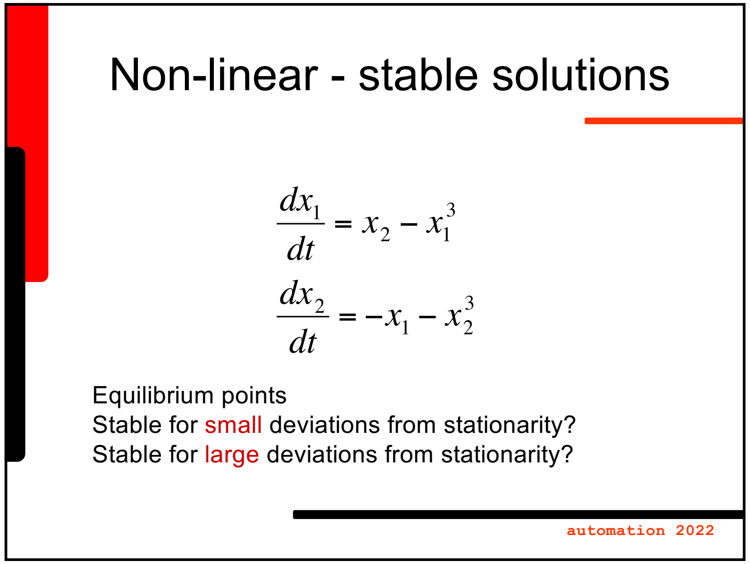


Non-linear - unstable solutions

$$\frac{dx_1}{dt} = x_2 + x_1^3$$

$$\frac{dx_2}{dt} = -x_1 + x_2^3$$

Equilibrium points
Stable for small deviations from stationarity?
Stable for large deviations from stationarity?

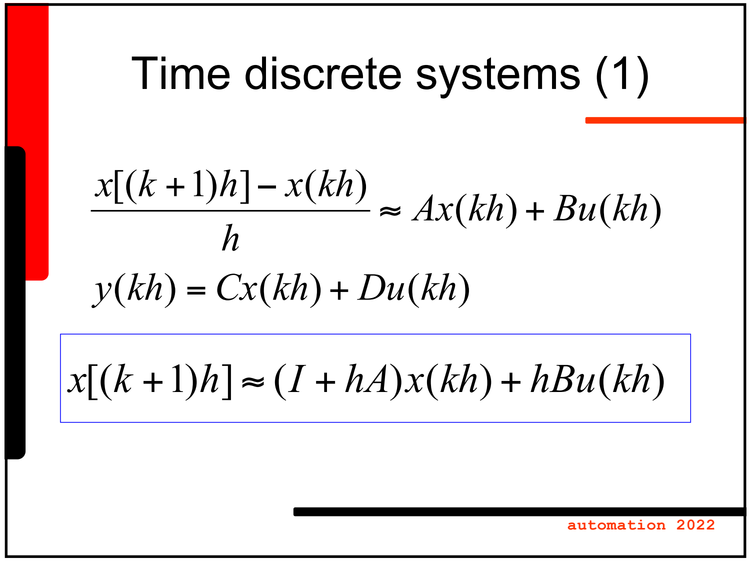


Non-linear - stable solutions

$$\frac{dx_1}{dt} = x_2 - x_1^3$$

$$\frac{dx_2}{dt} = -x_1 - x_2^3$$

Equilibrium points
Stable for small deviations from stationarity?
Stable for large deviations from stationarity?

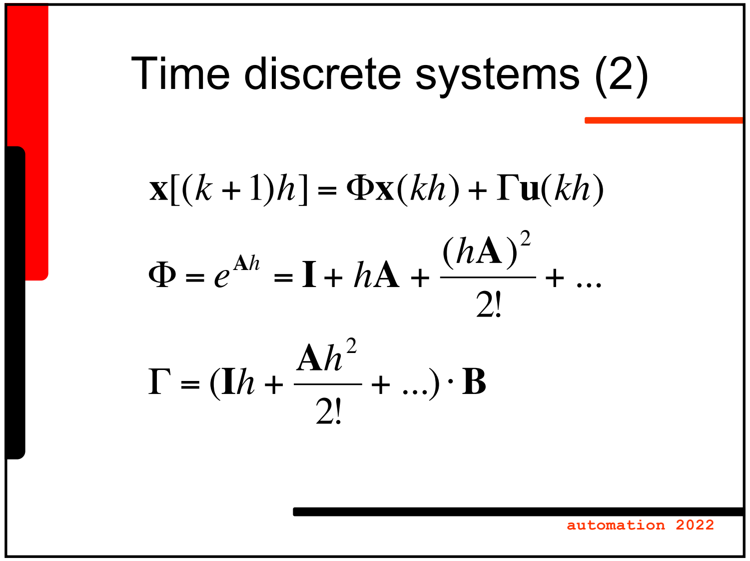


Time discrete systems (1)

$$\frac{x[(k+1)h] - x(kh)}{h} \approx Ax(kh) + Bu(kh)$$

$$y(kh) = Cx(kh) + Du(kh)$$

$$\boxed{x[(k+1)h] \approx (I + hA)x(kh) + hBu(kh)}$$

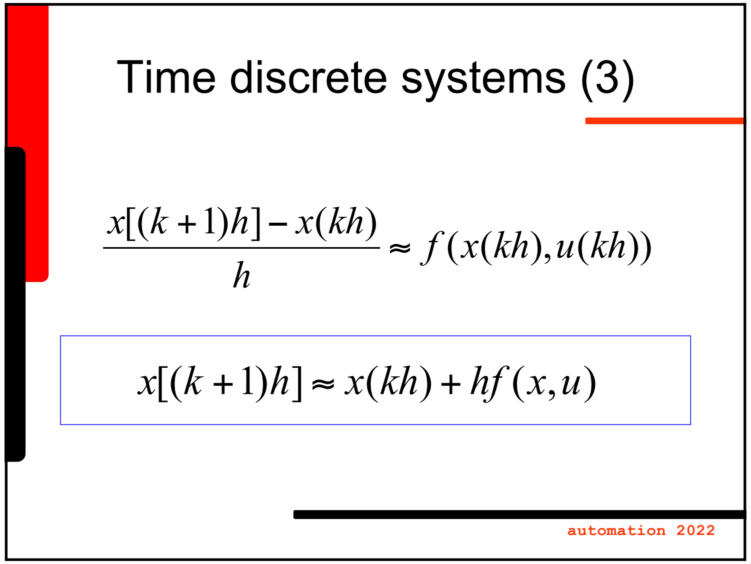


Time discrete systems (2)

$$\mathbf{x}[(k+1)h] = \Phi\mathbf{x}(kh) + \Gamma\mathbf{u}(kh)$$

$$\Phi = e^{\mathbf{A}h} = \mathbf{I} + h\mathbf{A} + \frac{(h\mathbf{A})^2}{2!} + \ldots$$

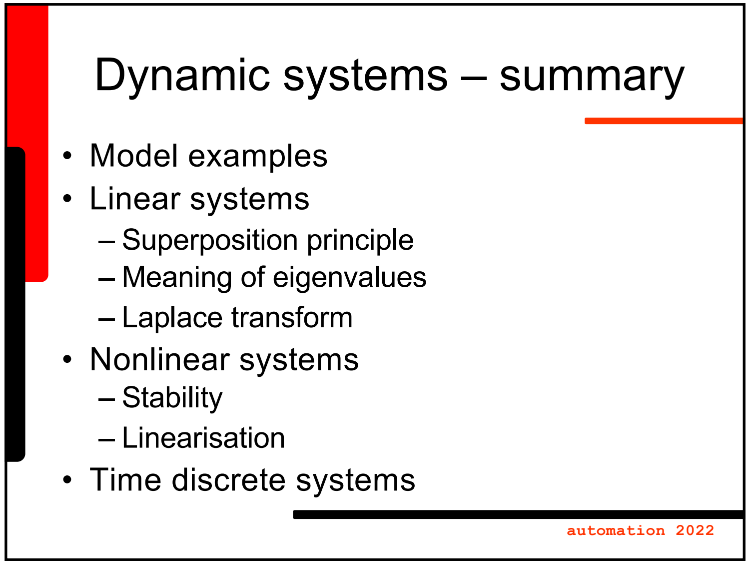
$$\Gamma = (\mathbf{I}h + \frac{\mathbf{A}h^2}{2!} + \ldots) \cdot \mathbf{B}$$



Time discrete systems (3)

$$\frac{x[(k+1)h]-x(kh)}{h} \approx f(x(kh),u(kh))$$

$$\boxed{x[(k+1)h] \approx x(kh) + hf(x,u)}$$



Dynamic systems – summary

- Model examples
- Linear systems
  - Superposition principle
  - Meaning of eigenvalues
  - Laplace transform
- Nonlinear systems
  - Stability
  - Linearisation
- Time discrete systems